

Representations of reduced enveloping algebras and cells in the affine Weyl group

Abstract: Let G be a semisimple algebraic group over an algebraically closed field of characteristic $p > 0$, and let \mathfrak{g} be its Lie algebra. The crucial Lie algebra representations to understand are those associated with the reduced enveloping algebra $U_\chi(\mathfrak{g})$ for a “nilpotent” $\chi \in \mathfrak{g}^*$. We conjecture that there is a natural assignment of simple modules in a typical block to left cells in the affine Weyl group W_p (for the dual root system) which lie in the two-sided cell corresponding to the orbit of χ in Lusztig’s bijection. This should respect the actions of the component group of $C_G(\chi)$ and fit naturally into Lusztig’s enriched bijection which also involves characters of $C_G(\chi)$. Some evidence will be described in special cases.

Representations of reduced enveloping algebras and cells in the affine Weyl group

- Cells in W_a
- Bijection between 2-sided cells and nilpotent orbits
- Lie algebra representations
- Assigning simple modules to left cells

Cells in affine Weyl group

W_a = affine Weyl group

Hecke algebra & KL polynomials
→ partition W_a into 2-sided cells

2-sided cell = disjoint union of left
cells (inverses form right cells)

elements of W_a \longleftrightarrow alcoves in euclidean space spanned by weight lattice X

dominant chamber = union of “canonical” left cells (Lusztig–Xi)

antidominant chamber = left cell in
“lowest” 2-sided cell

Lusztig's bijection

\mathcal{N} = nilpotent variety in \mathfrak{g} (p good)

nilpotent orbit $\mathcal{O}_e \longleftrightarrow$ 2-sided cell
in (dual) affine Weyl group \longleftrightarrow
canonical left cell C_e

\mathcal{B} = flag variety of G

\mathcal{B}_e = Springer fiber ($e \in \mathcal{N}$)

$A_e = C_G(e)/C_G(e)^\circ$ component
group

Conj: # left cells in 2-sided cell
corresp. to \mathcal{O}_e is $\dim H^*(\mathcal{B}_e)^{A_e}$

Lie algebra representations

G = simple algebraic group in char.
 $p > 0$, with $\mathfrak{g} = \text{Lie}(G)$

$U_\chi(\mathfrak{g}) = U(\mathfrak{g}) / \langle x^p - x^{[p]} - \chi(x)^p \rangle$
where $\chi \in \mathfrak{g}^*$

simple $U(\mathfrak{g})$ -module =
simple $U_\chi(\mathfrak{g})$ -module for some χ

Kac–Weisfeiler, Premet:

$2d = \dim G \cdot \chi \implies p^d$ divides all
dimensions of $U_\chi(\mathfrak{g})$ -modules (mild
restrictions on p , type of G)

Crucial case: χ “nilpotent”
(say $\chi \leftrightarrow e \in \mathcal{N}$)

Brown–Gordon:

blocks of $U_\chi(\mathfrak{g}) \longleftrightarrow W$ -orbits in X/pX (for dot action)

Friedlander–Parshall, Jantzen:

(standard Levi form) $e \in \mathcal{O}$ regular in Levi subalgebra of parabolic defined by set I of simple reflections

$\implies W/W_I$ parametrizes simple modules in typical block

Bezrukavnikov–Mirković–Rumynin:

(general case) $\#$ simples in typical block = $\dim H^*(\mathcal{B}_e)$, compatible with A_e -actions

special case: standard Levi form

Conjecture: Natural assignment of simples in typical block to left cells, compatible with Lusztig's enriched bijection

Special cases:

- restricted representations
- distinguished involution in canonical left cell \mathcal{C}_e
- standard Levi form
- A_e -orbit of simple modules \rightarrow single left cell