Cohomology for Frobenius kernels

by

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I. Introduction

G - a simple simply connected algebraic group scheme defined and split over the field $\mathbb{F}_p$ with p elements, considered as a group scheme over

k - the algebraic closure of $\mathbb{F}_p$

G_r - the rth Frobenius kernel ($G_r = \ker F^r : G \to G^{(r)}$)

Φ - root system

Φ^+ - positive roots

Φ^- - negative roots

Δ - simple roots

T - maximal split torus

B - Borel subgroup corresponding to the negative roots

U - unipotent radical of B

$X(T)$ - weights

$X(T)_+$ - dominant weights

$$\{\lambda \in X(T)\ :\ \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$$

$X_r(T)$ - p^r-restricted weights

$$\{\lambda \in X(T)\ :\ 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in \Delta\}$$

h - the Coxeter number of G

W - the Weyl group

G-modules:

Simples: $L(\lambda)$ for $\lambda \in X(T)_+$

Induced: $H^0(\lambda) = \operatorname{ind}_B^G \lambda$ for $\lambda \in X(T)_+$

λ - denotes the one dimensional B-module with trivial U action

G-socle of $H^0(\lambda)$ is $L(\lambda)$

Alternate notation: $\mathrm{H}^0(G/B, \mathcal{L}(\lambda))$ or $\nabla(\lambda)$

Motivating Theorem:

(Andersen-Jantzen 1984, Kumar-Lauritzen-Thomsen 1999)

Suppose $p > h$, $\gamma \in X(T)_+$, $w \in W$, and $w \cdot 0 + p\gamma \in X(T)_+$. Then

$$\mathrm{H}^i(G_1, H^0(w \cdot 0 + p\gamma)) \cong \begin{cases} \operatorname{ind}_B^G \left(S^{(i-\ell(w))/2}\mathfrak{u}^* \otimes \gamma\right)^{(1)} & \text{if } i - \ell(w) \text{ is even,} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathfrak{u} = \mathrm{Lie}(U)$.

Questions:

(1) What about small primes?

(2) What about G_r for $r > 1$?

Answers:

Andersen-Jantzen (1984) - $p = h$, $p = h - 1$, G_2 for $p = 3$
(the generic answer does not always hold)

Andersen (1987) - observed that $\mathrm{H}^1(G_r, H^0(\lambda))$ can be computed from the G_1-results

Jantzen (1991) - $\mathrm{H}^1(G_1, H^0(\lambda))$ for all $\lambda \in X(T)_+$ and all primes p.

Strategy:

- $\mathrm{H}^1(G_1, H^0(\lambda)) \cong \mathrm{ind}_B^G(\mathrm{H}^1(B_1, \lambda)^{(-1)})^{(1)}$
- $\mathrm{H}^1(B_1, \lambda) \cong (\mathrm{H}^1(U_1, k) \otimes \lambda)^{T_1}$
- $\mathrm{H}^1(U_1, k) \cong \mathrm{H}^1(\mathfrak{u}, k)$

Bendel-Nakano-Pillen (2004, current):

- $\mathrm{H}^1(G_r, H^0(\lambda))$ for all primes and all r
- $\mathrm{H}^2(G_r, H^0(\lambda))$ for $p \geq 3$ and all r

Note. $\mathrm{H}^2(G_1, M) \cong \mathrm{H}^2(u(\mathfrak{g}), M)$ for $\mathfrak{g} = \mathrm{Lie}(G)$ is the set of equivalence classes of central extensions of restricted Lie algebras:

$$0 \to M \to \mathfrak{h} \to \mathfrak{g} \to 0.$$

Ordinary Lie algebra cohomology:

$\mathrm{H}^\bullet(\mathfrak{u}, k)$ is the cohomology of the complex

$$k \xrightarrow{d_0} \mathfrak{u}^* \xrightarrow{d_1} \Lambda^2(\mathfrak{u})^* \xrightarrow{d_2} \Lambda^3(\mathfrak{u})^* \to \cdots$$

$d_0 = 0$

$d_1 : \mathfrak{u}^* \to \Lambda^2(\mathfrak{u})^*$ by

$$(d_1\phi)(x \wedge y) = -\phi([x, y])$$

for $\phi \in \mathfrak{u}^*$, $x, y \in \mathfrak{u}$

d_i for $i > 1$ can be determined by a product rule

Correlated Theorem:

(Friedlander-Parshall 1987, Polo-Tilouine 2003)

Suppose $p \geq h - 1$. As a T-module,

$$\mathrm{H}^i(\mathfrak{u}, k) \cong \bigoplus_{w \in W,\ \ell(w) = i} -w \cdot 0.$$

Note: originally due to Kostant (1961) in characteristic 0.

Question: What about small primes?

Answers: For $p > 3$, the generic answer holds for H^1 and H^2

Jantzen (1991) - $\mathrm{H}^1(\mathfrak{u}, k) \cong \bigoplus_{\alpha \in \Delta} \alpha$ for $p > 3$

BNP - $\mathrm{H}^2(\mathfrak{u}, k) \cong \bigoplus_{\ell(w)=2} -w \cdot 0$ for $p > 3$

Extra classes can arise for $p = 2, 3$.

II. First Cohomology:

$w \in W$ with $\ell(w) = 1$:

- $w = s_\alpha$ for $\alpha \in \Delta$
- $s_\alpha \cdot 0 = -\alpha$

Original Theorem. (AJ,KLT) Suppose $p > h$ and $\lambda \in X(T)_+$.

$$\mathrm{H}^1(G_1, H^0(\lambda)) \cong \begin{cases} H^0(\gamma)^{(1)} & \text{if } \lambda = p\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem. (Jantzen)

- $p > 3$: the same generic answer holds
- $p = 3$: there are non-generic answers in types A_2 and G_2
- $p = 2$: there are non-generic answers in types A_3, B_n $(n \geq 3)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$, F_4, and G_2.

For $r > 1$: Use the Lyndon-Hochschild-Serre (LHS) spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(B_r/B_1, \mathrm{H}^j(B_1, \lambda)) \Rightarrow \mathrm{H}^{i+j}(B_r, \lambda).$$

Theorem. (BNP) Let $p > 3$ and $\lambda \in X(T)$. Then

$$\mathrm{H}^1(B_r, \lambda) \cong \begin{cases} \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^i\alpha \text{ for } \gamma \in X(T), \alpha \in \Delta, 0 \leq i \leq r-1 \\ 0 & \text{else.} \end{cases}$$

Again, there are non-generic answers for

- $p = 3$ in types A_2 and G_2
- $p = 2$ in types A_3, B_n $(n \geq 3)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$, F_4, and G_2.

Question: Where does the non-genericity come from?

1. Repeated Weights/Doubling of Cohomology

Example: Consider type A_2 with simple roots α_1 and α_2. Let ω_1 and ω_2 denote the corresponding fundamental dominant weights. Notice that for $p = 3$

$$p\omega_1-\alpha_1 = 3\omega_1-(2\omega_1-\omega_2) = \omega_1+\omega_2 = 3\omega_2-(-\omega_1+2\omega_2) = p\omega_2-\alpha_2.$$

And we find that

$$\mathrm{H}^1(B_r, p^{r-1}(\omega_1+\omega_2)) \cong \omega_1^{(r)} \oplus \omega_2^{(r)}.$$

Whenever we have "repeated" weights like this, we get a "doubling" of the cohomology.

2. Extra Cohomology

Example. Consider type G_2 and $p = 3$. Let α_1 and α_2 denote the simple roots with α_2 the long root.

As a T-module, $\mathrm{H}^1(\mathfrak{u}, k) \cong \alpha_1 \oplus \alpha_2 \oplus (3\alpha_1+\alpha_2)$.

And as a B-module, $\mathrm{H}^1(\mathfrak{u}, k) \cong \alpha_1 \oplus M_{G_2}$

where M_{G_2} is a two dimensional indecomposable B-module with head $3\alpha_1+\alpha_2$ and socle α_2.

$\mathrm{H}^1(B_r, p^r\gamma - p^{r-1}\alpha_2) \cong (N_{G_2} \otimes \gamma)^{(r)}$

where N_{G_2} is a two dimensional indecomposable B-module with head α_1 and socle k.

Compare to $\mathrm{H}^1(B_r, p^r\gamma - p^{r-1}\alpha_1) \cong \gamma^{(r)}$.

In general, extra classes in $\mathrm{H}^1(\mathfrak{u}, k)$ leads to the presence of indecomposable B-modules.

G_r-cohomology: Once the B_r-cohomology is known we can use the isomorphism

$$\mathrm{H}^1(G_r, H^0(\lambda)) \cong \mathrm{ind}_B^G \left(\mathrm{H}^1(B_r, \lambda)^{(-r)}\right)^{(r)}$$

to obtain the G_r-cohomology of induced modules.

Theorem A. (Andersen, BNP) Let $p > 3$ and $\lambda \in X(T)_+$. Then

$$\mathrm{H}^1(G_r, H^0(\lambda)) \cong \begin{cases} H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^i\alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r-1 \\ 0 & \text{else.} \end{cases}$$

Theorem B. (BNP) Let $p = 3$ and $\lambda \in X(T)_+$.

(a) G is not of type A_2 or G_2. Then

$$\mathrm{H}^1(G_r, H^0(\lambda)) \cong \begin{cases} H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^i\alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r-1 \\ 0 & \text{else.} \end{cases}$$

(b) G is of type A_2. Then

$$\mathrm{H}^1(G_r, H^0(\lambda)) \cong \begin{cases} H^0(\gamma+\omega_1)^{(r)} \oplus H^0(\gamma+\omega_2)^{(r)} & \begin{aligned} \text{if } \lambda &= p^r\gamma + p^{r-1}(\omega_1+\omega_2) \\ &= p^r(\gamma-\omega_1) - p^{r-1}\alpha_1 \\ &= p^r(\gamma-\omega_2) - p^{r-1}\alpha_2, \end{aligned} \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^i\alpha \text{ for } \alpha \in \Delta, \\ & \qquad 0 \leq i \leq r-2, \\ 0 & \text{else.} \end{cases}$$

(c) G is of type G_2. Then
$\mathrm{H}^1(G_r, H^0(\lambda)) \cong$

$$\begin{cases} H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^{r-1}\alpha_1 \\ \operatorname{ind}_B^G(N_{G_2} \otimes \gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^{r-1}\alpha_2, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^i\alpha \text{ for } \alpha \in \Delta, 0 \le i \le r-2, \\ 0 & \text{else.} \end{cases}$$

$N_{G_2} \otimes \gamma$ has factors $\gamma + \alpha_1$ and γ.

If $\lambda = p^r\gamma - p^{r-1}\alpha_2 \in X(T)_+$, then

- $\langle \gamma, \alpha_1^\vee \rangle \ge -1$
- $\langle \gamma, \alpha_2^\vee \rangle \ge 1$

So $\gamma + \alpha_1$ is dominant but γ need not be.

Case 1. $\langle \gamma, \alpha_1^\vee \rangle = -1$

$$\operatorname{ind}_B^G(N_{G_2} \otimes \gamma) \cong H^0(\gamma + \alpha_1)$$

Case 2. $\langle \gamma, \alpha_1^\vee \rangle \ge 0$

$\operatorname{ind}_B^G(N_{G_2} \otimes \gamma)$ has a filtration with factors $H^0(\gamma + \alpha_1)$ and $H^0(\gamma)$ (from top to bottom).

For $p = 2$, things are similar. In all cases, one can get an explicit good filtration for $\mathrm{H}^1(G_r, H^0(\lambda))^{(-r)}$.

III. Applications of First Cohomology Results

(1) Extension Formulas:

Theorem. (BNP) Let $\lambda, \mu \in X_r(T)$ and $p \geq 2(h-1)$. Then $\operatorname{Ext}^1_{G_r}(L(\lambda), L(\mu))$ is a semisimple G-module and

$$\operatorname{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \pi_h} \operatorname{Ext}^1_G(L(\lambda), L(\nu)^{(r)} \otimes L(\mu)) \otimes L(\nu)^{(r)}$$

where

$$\pi_h = \{\nu \in X(T)_+ : \langle \nu, \alpha_0^\vee \rangle < h\}$$

Note: See C. Pillen's talk.

(2) B_r to G_r:

Theorem. (BNP) Let p be an arbitrary prime and $\lambda \in X(T)_+$. Then

$$\mathrm{H}^2(G_r, H^0(\lambda)) \cong \operatorname{ind}_B^G(\mathrm{H}^2(B_r, \lambda)^{(-r)})^{(r)}.$$

Proof: Use the spectral sequence

$$E_2^{i,j} = R^i \operatorname{ind}_B^G \left(\mathrm{H}^j(B_r, \lambda)^{(-r)}\right) \Rightarrow \mathrm{H}^{i+j}(G_r, \operatorname{ind}_B^G \lambda)^{(-r)}.$$

IV. Second Cohomology

Case 1. $\ell(w) = 0$

Orginal Theorem. Suppose $p > h$ and $\gamma \in X(T)_+$. Then

- $\mathrm{H}^2(G_1, k) \cong \mathrm{ind}_B^G(\mathfrak{u}^*)^{(1)}$.
- $\mathrm{H}^2(G_1, H^0(p\gamma)) \cong \mathrm{ind}_B^G(\mathfrak{u}^* \otimes \gamma)^{(1)}$.

Case 2. $\ell(w) = 2$

- $w = s_\alpha s_\beta$ for $\alpha, \beta \in \Delta$ and $\alpha \neq \beta$
- $w{\cdot}0 = (\langle \beta, \alpha^\vee \rangle - 1)\alpha - \beta = m\alpha - \beta$ where $m \in \{-1, -2, -3, -4\}$

Original Theorem. Suppose $p > h$, $\lambda \in X(T)_+$, and $\lambda \notin pX(T)$. Then

$$\mathrm{H}^2(G_1, H^0(\lambda)) \cong \begin{cases} H^0(\gamma)^{(1)} & \text{if } \lambda = w \cdot 0 + p\gamma \text{ for } w \in W \text{ with } \ell(w) = 2, \\ 0 & \text{else.} \end{cases}$$

Strategy for small primes:

- $\mathrm{H}^2(G_r, H^0(\lambda)) \cong \mathrm{ind}_B^G(\mathrm{H}^2(B_r, \lambda)^{(-r)})^{(r)}$
- For $r > 1$, use LHS spectral sequence to go from $\mathrm{H}^2(B_1, \lambda)$ to higher $\mathrm{H}^2(B_r, \lambda)$.
- $\mathrm{H}^2(B_1, \lambda) \cong \mathrm{H}^2(U_1, \lambda)^{T_1} \cong (\mathrm{H}^2(U_1, k) \otimes \lambda)^{T_1}$
- For $p \geq 3$, use the spectral sequence of Friedlander-Parshall:

$$E_2^{2i,j} = S^i(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{u}, k) \Rightarrow \mathrm{H}^{i+j}(U_1, k).$$

- Compute $\mathrm{H}^2(\mathfrak{u}, k)$.

V. $\mathrm{H}^2(\mathfrak{u}, k)$

Goal: Show that almost all weights are of the form $-w \cdot 0$ with $w \in W$ and $\ell(w) = 2$.

Lemma. If $p = 3$, assume that Φ is not of type G_2. If γ is a weight of $\mathrm{H}^2(\mathfrak{u}, k)$, then

$$\gamma = \alpha + \beta$$

where $\alpha \in \Delta$ and $\beta \in \Phi^+$, $\alpha \neq \beta$.

For such pairs α, β, we are led to consider whether there exist $\sigma \in X(T)$, $\beta_1, \beta_2 \in \Delta$, $0 < i < p$, and $m \geq 0$ such any of the following hold:

Equation 1. $\alpha + \beta = p\sigma$

Equation 2. $\alpha + \beta = \beta_1 + p\sigma$

Equation 3. $\alpha + \beta = i\beta_1 + p^m\beta_2 + p\sigma$

Suppose $\gamma = \alpha + \beta$ is a weight of $\mathrm{H}^2(\mathfrak{u}, k)$.

Fact 1. If γ does not satisfy Equation (2), then $\mathrm{H}^2(B_1, -\gamma) \neq 0$.

Fact 2. If γ does not satisfy Equations (1) and (2), then $\mathrm{H}^2(B, -\gamma + p\nu) \neq 0$ for some $\nu \in X(T)$. From that it follows that γ must satisfy Equation (3).

Note. $\gamma = -w \cdot 0 = i\beta_1 + \beta_2$ satisfies Equation (3) with $m = 0$ and $\sigma = 0$.

Solutions: (omitting type G_2 when $p = 3$)

Equation 1.

- $p > 3$: no solutions
- $p = 3$: Solutions in type A_2
  - $\alpha_1 + (\alpha_1 + \alpha_2) = -(s_{\alpha_1} s_{\alpha_2}) \cdot 0 = 3\omega_1$
  - $\alpha_2 + (\alpha_1 + \alpha_2) = -(s_{\alpha_2} s_{\alpha_1}) \cdot 0 = 3\omega_2$

Equation 2.

- $p > 3$: no solutions
- $p = 3$: Solutions in types
  - B_n: $\alpha_n + (\alpha_{n-1} + 2\alpha_n) = -(s_{\alpha_n} s_{\alpha_{n-1}}) \cdot 0$
  - C_n: $\alpha_{n-1} + (2\alpha_{n-1} + \alpha_n) = -(s_{\alpha_{n-1}} s_{\alpha_n}) \cdot 0$
  - F_4: $\alpha_3 + (\alpha_2 + 2\alpha_3) = -(s_{\alpha_3} s_{\alpha_2}) \cdot 0$

Equation 3. Numerous solutions. Restrict attention to weights $\alpha + \beta$ which actually arise as weights of $\mathrm{H}^2(\mathfrak{u}, k)$. In addition to the above, we find two types of solutions.

Type 1: Weights $\alpha + \beta = -w \cdot 0$ with $w \in W$ and $\ell(w)$ where there exists $w' \in W$ with $\ell(w') = 2$ and $w \cdot 0 = w' \cdot 0 + p\sigma$ for some $\sigma \in X(T)$.

This occurs for

- $p = 5$ in type A_4
- $p = 3$ in type A_5
- $p = 3$ in type E_6

Type 2: Weights not of the form $-w \cdot 0$.

This occurs only for $p = 3$ in types $B_n (n \geq 3)$, $C_n (n \geq 3)$, and F_4.

Theorem. (Friedlander-Parshall, Polo-Tilouine, BNP)
Let $p \geq 3$ and $\pi = \{-w \cdot 0 \ : \ w \in W, \ \ell(w) = 2\}$. As a T-module

$$\mathrm{H}^2(\mathfrak{u}, k) \cong \bigoplus_{\lambda \in \pi \cup \pi'} \lambda$$

where π' is given below.

Further, if $\lambda = -w \cdot 0$ with $w = s_\alpha s_\beta$, then the corresponding cohomology class is represented by $\phi_\alpha \wedge \phi_{-\langle \beta, \alpha^\vee \rangle \alpha + \beta} \in \Lambda^2(\mathfrak{u}^*)$.

Here, $\phi_\sigma \in \mathfrak{u}^*$ is dual to $x_{-\sigma} \in \mathfrak{u}$ for $\sigma \in \Phi^+$.

(a) $p > 3$ or Φ is of type A_n, D_n, or E_n: $\pi' = \emptyset$.

(b) Φ is of type B_n, $n \geq 3$: $\pi' = \{\alpha_{n-2} + 2\alpha_{n-1} + 3\alpha_n\}$ corresponding to the cohomology class

$$\phi_{\alpha_n} \wedge \phi_{\alpha_{n-2}+2\alpha_{n-1}+2\alpha_n} - \phi_{\alpha_{n-1}+\alpha_n} \wedge \phi_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n} + \phi_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n} \wedge \phi_{\alpha_{n-1}+2\alpha_n}.$$

(c) Φ is of type C_n, $n \geq 3$: $\pi' = \{\alpha_{n-2} + 3\alpha_{n-1} + \alpha_n\}$ corresponding to the cohomology class

$$\phi_{\alpha_{n-1}} \wedge \phi_{\alpha_{n-2}+2\alpha_{n-1}+\alpha_n} - \phi_{\alpha_{n-2}+\alpha_{n-1}} \wedge \phi_{2\alpha_{n-1}+\alpha_n}.$$

(d) Φ is of type F_4: $\pi' = \{\alpha_1 + 2\alpha_2 + 3\alpha_3, \alpha_2 + 3\alpha_3 + \alpha_4\}$ corresponding to the cohomology classes

$$\phi_{\alpha_3} \wedge \phi_{\alpha_1+2\alpha_2+2\alpha_3} - \phi_{\alpha_2+\alpha_3} \wedge \phi_{\alpha_1+\alpha_2+2\alpha_3} + \phi_{\alpha_1+\alpha_2+\alpha_3} \wedge \phi_{\alpha_2+2\alpha_3},$$
$$\phi_{\alpha_3} \wedge \phi_{\alpha_2+2\alpha_3+\alpha_4} - \phi_{\alpha_3+\alpha_4} \wedge \phi_{\alpha_2+2\alpha_3}.$$

(e) Φ is of type G_2: $\pi' = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 3\alpha_2, 6\alpha_1 + 3\alpha_2, 4\alpha_1 + 2\alpha_2\}$ corresponding to the cohomology classes

$$\phi_{\alpha_1} \wedge \phi_{2\alpha_1+\alpha_2}, \phi_{\alpha_2} \wedge \phi_{3\alpha_1+2\alpha_2}, \phi_{3\alpha_1+\alpha_2} \wedge \phi_{3\alpha_1+2\alpha_2},$$
$$\phi_{\alpha_1} \wedge \phi_{3\alpha_1+2\alpha_2} + \phi_{\alpha_1+\alpha_2} \wedge \phi_{3\alpha_1+\alpha_2}.$$

VI. $\mathrm{H}^2(B_1, \lambda)$

Case 1. $\mathrm{H}^2(B_1, k)$

- $\mathrm{H}^2(B_1, k) \cong \mathrm{H}^2(U_1, k)^{T_1}$
- As T-modules, $\mathrm{H}^2(U_1, k) \cong (\text{ submodule of } \mathrm{H}^2(\mathfrak{u}, k)) \oplus (\mathfrak{u}^*)^{(1)}$

Theorem. (BNP) Suppose $p \geq 3$. As B-modules,

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)}.$$

except for

- (Andersen-Jantzen) $p = 3$ in type A_2:

  $$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus \omega_1^{(1)} \oplus \omega_2^{(1)}.$$

- (Andersen-Jantzen) $p = 3$ in type G_2: there is a non-split exact sequence

  $$0 \to (\mathfrak{u}^*)^{(1)} \to \mathrm{H}^2(B_1, k) \to M_{G_2}^{(1)} \to 0$$

  where M_{G_2} is an indecomposable B-module with head ω_1 and socle $\omega_2 - \omega_1$.

Case 2. $\mathrm{H}^2(B_1, \lambda)$ with $\lambda \notin pX(T)$

$\mathrm{H}^1(B_1, \lambda) \cong (\mathrm{H}^1(U_1, k) \otimes \lambda)^{T_1}$

Then

$\mathrm{H}^2(B_1, \lambda) \subset (\mathrm{H}^2(\mathfrak{u}, k) \otimes \lambda)^{T_1}$

Theorem. (BNP) Suppose $p \geq 3$ and $\lambda \in X(T)$ with $\lambda \notin pX(T)$.

If $\mathrm{H}^2(B_1, \lambda) \neq 0$, then $\lambda = w \cdot 0 + p\gamma$ for some $\gamma \in X(T)$ and $w \in W$ with $\ell(w) = 2$.

If $\lambda = w \cdot 0 + p\gamma$, then generically

$$\mathrm{H}^2(B_1, \lambda) \cong \gamma^{(1)}.$$

Non-generic answers:

- If $w \cdot 0 = w' \cdot 0 + p\sigma$ - see Equation (3), Type 1.
- Extra weights in $\mathrm{H}^2(\mathfrak{u}, k)$ - not of the form $-w \cdot 0$
  - see also Equation (3), Type 2.
- Solutions to Equation (2).
  - may not yield weights of $\mathrm{H}^2(U_1, k)$

Doubling: (Assume $w \cdot 0 + p\gamma \in X_1(T)$.)

(i) $p = 5$, Φ is of type A_4, and $w \in \{s_{\alpha_2}s_{\alpha_1}, s_{\alpha_3}s_{\alpha_4}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = \mathrm{H}^2(B_1, 2\omega_2 + 2\omega_3) \cong \omega_2^{(1)} \oplus \omega_3^{(1)}.$$

(ii) $p = 3$, Φ is of type A_5, and $w \in \{s_{\alpha_1}s_{\alpha_2}, s_{\alpha_5}s_{\alpha_4}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = \mathrm{H}^2(B_1, \omega_3) \cong \omega_1^{(1)} \oplus \omega_5^{(1)}.$$

(iii) $p = 3$, Φ is of type A_5, and $w \in \{s_{\alpha_2}s_{\alpha_1}, s_{\alpha_4}s_{\alpha_5}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = \mathrm{H}^2(B_1, 2\omega_3) \cong \omega_2^{(1)} \oplus \omega_4^{(1)}.$$

(iv) $p = 3$, Φ is of type A_5, and $w \in \{s_{\alpha_1}s_{\alpha_4}, s_{\alpha_2}s_{\alpha_5}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = \mathrm{H}^2(B_1, \rho) \cong (\omega_1 + \omega_4)^{(1)} \oplus (\omega_2 + \omega_5)^{(1)}.$$

(v) $p = 3$, Φ is of type E_6, and $w \in \{s_{\alpha_1}s_{\alpha_3}, s_{\alpha_6}s_{\alpha_5}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = \mathrm{H}^2(B_1, \omega_4) \cong \omega_1^{(1)} \oplus \omega_6^{(1)}.$$

(vi) $p = 3$, Φ is of type E_6, and $w \in \{s_{\alpha_3}s_{\alpha_1}, s_{\alpha_5}s_{\alpha_6}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = \mathrm{H}^2(B_1, 2\omega_4) \cong \omega_3^{(1)} \oplus \omega_5^{(1)}.$$

(vii) $p = 3$, Φ is of type E_6, and $w \in \{s_{\alpha_1}s_{\alpha_5}, s_{\alpha_3}s_{\alpha_6}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = \mathrm{H}^2(B_1, \rho - \omega_2) \cong (\omega_1 + \omega_5)^{(1)} \oplus (\omega_3 + \omega_6)^{(1)}.$$

Vanishing:

(viii) $p = 3$, Φ is of type B_n, $n > 2$, and $w = s_{\alpha_n} s_{\alpha_{n-1}}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = 0.$$

(ix) $p = 3$, Φ is of type C_n, and $w = s_{\alpha_{n-1}} s_{\alpha_n}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = 0.$$

(x) $p = 3$, Φ is of type F_4, and $w = s_{\alpha_3} s_{\alpha_2}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = 0.$$

Indecomposables:

(xi) $p = 3$, Φ is of type B_n, $n > 2$, and $w = s_{\alpha_{n-1}} s_{\alpha_{n-2}}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) \cong (N_{B_n} \otimes \gamma)^{(1)}.$$

(xii) $p = 3$, Φ is of type C_n, $n > 2$, and $w = s_{\alpha_{n-2}} s_{\alpha_n}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma) = (N_{C_n} \otimes \gamma)^{(1)}.$$

(xiii) $p = 3$, Φ is of type F_4, and $w \in \{s_{\alpha_2} s_{\alpha_1}, s_{\alpha_2} s_{\alpha_4}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) \cong (N_{F_4} \otimes \gamma)^{(1)}.$$

(xiv) $p = 3$, Φ is of type G_2, and $w = s_{\alpha_2} s_{\alpha_1}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) \cong (N_{G_2} \otimes \gamma)^{(1)}.$$

The modules N_{X_n} are all two dimensional indecomposable B-modules with factors from top to bottom as follows:

- N_{B_n} : α_n and k
- N_{C_n}: α_{n-1} and k
- N_{F_4}: α_3 and k
- N_{G_2}: α_1 and k

VII. $\mathrm{H}^2(B_r, \lambda)$

For higher r, we use the LHS spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(B_r/B_1, \mathrm{H}^j(B_1, \lambda)) \Rightarrow \mathrm{H}^{i+j}(B_r, \lambda).$$

Case 1. $\lambda \in p^r X(T)$

Proposition. Suppose $p \geq 3$. Then

$$\mathrm{H}^2(B_r, k) \cong \mathrm{H}^2(B_1, k)^{(r-1)}.$$

Case 2. $\lambda \notin p^r X(T)$

Lemma. Suppose $p \geq 3$, $0 \leq l < r$, and $\alpha \in \Delta$.

(a) Then

$$\mathrm{H}^2(B_r, -p^l\alpha) \cong \begin{cases} k & \text{if } l > 0, \\ 0 & \text{if } l = 0. \end{cases}$$

(b) Suppose $p = 3$ and Φ is of type A_2. Then for $i \in \{1, 2\}$,

$$\mathrm{H}^2(B_r, -p^l\omega_i) \cong \begin{cases} k & \text{if } l > 0, \\ 0 & \text{if } l = 0. \end{cases}$$

For $p = 3$ in type G_2, also need information about $\mathrm{H}^1(B_r, N_{G_2})$.

Theorem. Let $p \geq 3$ and $\lambda \in X(T)$.
(a) If $p \neq 3$ or Φ is not of type A_2 or G_2 then

$$\mathrm{H}^2(B_r, \lambda) \cong \begin{cases} \mathrm{H}^2(B_1, w \cdot 0 + p\gamma)^{(r-1)} & \text{if } \lambda = p^{r-1}(w \cdot 0 + p\gamma) \\ & \quad \text{with } \ell(w) = 2 \text{ or } 0, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma + p^l w \cdot 0, \\ & \text{ with } \ell(w) = 2,\ 0 \leq l < r-1, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \\ & \quad \text{with } 0 < l < r \text{ and } \alpha \in \Delta, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \\ & \quad \text{with } 0 \leq l < k < r\text{and } \alpha, \beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(b) If $p = 3$ and Φ is of type A_2, then

$$\mathrm{H}^2(B_r, \lambda) \cong \begin{cases} (\mathfrak{u}^{*(r)} \oplus \omega_1^{(r)} \oplus \omega_2^{(r)}) \otimes \gamma^{(r)} & \text{if } \lambda = p^r\gamma, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^l w \cdot 0, \\ & \text{ with } \ell(w) = 2,\ 0 \leq l < r-1, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \\ & \quad \text{with } 0 < l < r \text{ and } \alpha \in \Delta, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \\ & \quad \text{with } 0 \leq l < k < r-1\text{and } \alpha, \beta \in \Delta, \\ (\gamma + \omega_1)^{(r)} \oplus (\gamma + \omega_2)^{(r)} & \text{if } \lambda = p^r\gamma + p^{r-1}(\omega_1 + \omega_2) - p^l\alpha, \\ & \text{ with } 0 \leq l < r-1 \text{ and } \alpha \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(c) If $p = 3$ and Φ is of type G_2, then

$$\mathrm{H}^2(B_r, \lambda) \cong \begin{cases} \mathrm{H}^2(B_1, w \cdot 0 + p\gamma)^{(r-1)} & \text{if } \lambda = p^{r-1}(p\gamma + w.0) \\ & \quad \text{with } \ell(w) = 2 \text{ or } 0, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma + p^l w.0, \\ & \quad \text{with } \ell(w) = 2, \text{and } 0 \leq l < r-1, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \\ & \quad \text{with } 0 < l < r \text{ and } \alpha \in \Delta, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \\ & \quad \text{with } 0 \leq l < k < r \text{ and } \alpha, \beta \in \Delta, \\ & \quad \text{where } k \neq r-1 \text{ if } \beta = \alpha_2, \text{ and} \\ & \quad k \neq l+1 \text{ if } \beta = \alpha_1 \text{ and } \alpha = \alpha_2, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^{l+1}(\beta + \alpha_1) - p^l\alpha_2, \\ & \quad \text{with } 0 \leq l < r-1 \text{and } \beta \in \Delta, \\ N_{G_2}^{(r)} \otimes \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^{r-1}\alpha_2 - p^l\alpha, \\ & \quad \text{with } 0 \leq l < r-1, \text{and } \alpha \in \Delta, \\ 0 & \text{else.} \end{cases}$$

VIII. $\mathrm{H}^2(B,\lambda)$.

Theorem. (Cline-Parshall-Scott)

Suppose $\lambda \in X(T)$. Then

$$\mathrm{H}^2(B,\lambda) \cong \varprojlim \mathrm{H}^2(B_r,\lambda).$$

Theorem. (O'Halloran, Andersen, BNP)

Let $p \geq 3$ and $\lambda \in X(T)$.

(a) If $p = 3$ assume that Φ is not of type G_2. Then

$$\mathrm{H}^2(B,\lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w.0, \text{ with } 0 \leq l, \ \ell(w) = 2, \\ k & \text{if } \lambda = -p^l\alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^k\beta - p^l\alpha, \text{ with } 0 \leq l < k \text{ and } \alpha, \beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(b) Suppose $p = 3$ and Φ is of type G_2. Then

$$\mathrm{H}^2(B,\lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w.0, \text{ with } 0 \leq l, \ \ell(w) = 2, \\ k & \text{if } \lambda = -p^l\alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^k\beta - p^l\alpha, \text{ with } 0 \leq l < k \text{ and } \alpha, \beta \in \Delta, \\ & \quad \text{where } k \neq l+1 \text{ if } \beta = \alpha_1 \text{ and } \alpha = \alpha_2, \\ k & \text{if } \lambda = -p^{l+1}(\beta + \alpha_1) - p^l\alpha_2, \text{ with } 0 \leq l \text{ and } \beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

IX. $\mathrm{H}^2(G_1, H^0(\lambda))$

Theorem. (Andersen-Jantzen, Kumar-Lauritzen-Thomsen, BNP)

Let $p \geq 3$ and $\lambda \in X(T)_+$.

(a) If $\lambda = p\gamma$, then $\mathrm{H}^2(G_1, H^0(\lambda)) \cong \mathrm{ind}_B^G(\mathfrak{u}^* \otimes \gamma)^{(1)}$ except in the following cases:

(i) $p = 3$, Φ is of type A_2, where

$\mathrm{H}^2(G_1, H^0(\lambda)) \cong \mathrm{ind}_B^G(\mathfrak{u}^* \otimes \gamma)^{(1)} \oplus H^0(\omega_1 + \gamma)^{(1)} \oplus H^0(\omega_2 + \gamma)^{(1)}$.

(ii) $p = 3$, Φ is of type G_2, where

$$\mathrm{H}^2(G_1, H^0(\lambda)) \cong \mathrm{ind}_B^G(\mathrm{H}^1(B_1, k)^{(-1)} \otimes \gamma)^{(1)}.$$

(b) If $\lambda \notin pX(T)$ and $\mathrm{H}^2(G_1, H^0(\lambda)) \neq 0$, then $\lambda = w \cdot 0 + p\gamma$ for some $w \in W$ with $\ell(w) = 2$ and $\gamma \in X(T)$.

(c) If $p \geq 5$ and $\lambda = w \cdot 0 + p\gamma$ for some $w \in W$ with $\ell(w) = 2$ and $\gamma \in X(T)$, then

$$\mathrm{H}^2(G_1, H^0(\lambda)) \cong H^0(\gamma)^{(1)},$$

except in the following case:

$p = 5$, Φ is of type A_4, and $w \in \{s_{\alpha_2}s_{\alpha_1}, s_{\alpha_3}s_{\alpha_4}\}$, where

$$\mathrm{H}^2(G_1, H^0(\lambda)) \cong H^0(\gamma)^{(1)} \oplus H^0(\gamma - \omega_2 + \omega_3)^{(1)}.$$

(d) If $p = 3$, one has many non-generic cases including doubling and vanishing of cohomology.

And some answers involve $\mathrm{ind}_B^G(N_{X_n} \otimes \gamma)$. In all cases, this module has a good filtration which can be explicitly determined depending on γ.

X. $\mathrm{H}^2(G_r, H^0(\lambda))$ **Theorem.** Let $p \geq 3$, $r > 1$, and $\lambda \in X(T)_+$.

(a) If $p \neq 3$ or Φ is not of type A_2 or G_2 then

$$\mathrm{H}^2(G_r, H^0(\lambda)) \cong \begin{cases} \operatorname{ind}_B^G(\mathrm{H}^2(B_1, w\cdot 0 + p\gamma))^{(r-1)} & \text{if } \lambda = p^{r-1}(w\cdot 0 + p\gamma) \\ & \quad \text{with } \ell(w) = 2 \text{ or } 0, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma + p^l w\cdot 0, \\ & \quad \text{with } \ell(w) = 2,\ 0 \leq l < r-1, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \\ & \quad \text{with } 0 < l < r \text{ and } \alpha \in \Delta, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \\ & \quad \text{with } 0 \leq l < k < r \text{ and } \alpha, \beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(b) If $p = 3$ and Φ is of type A_2, then

$$\mathrm{H}^2(G_r, H^0(\lambda)) \cong \begin{cases} \operatorname{ind}_B^G(\mathfrak{u}^* \oplus \omega_1 \oplus \omega_2)^{(r)} \otimes \gamma^{(r)} & \text{if } \lambda = p^r\gamma, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^l w\cdot 0, \\ & \quad \text{with } \ell(w) = 2,\ 0 \leq l < r-1, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \\ & \quad \text{with } 0 < l < r \text{ and } \alpha \in \Delta, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \\ & \quad \text{with } 0 \leq l < k < r-1 \\ & \quad \text{and } \alpha, \beta \in \Delta, \\ H^0(\gamma + \omega_1)^{(r)} \oplus H^0(\gamma + \omega_2)^{(r)} & \text{if } \lambda = p^r\gamma + p^{r-1}(\omega_1 + \omega_2) - p^l\alpha, \\ & \quad \text{with } 0 \leq l < r-1 \text{ and } \alpha \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(c) If $p = 3$ and Φ is of type G_2, then
$\mathrm{H}^2(G_r, H^0(\lambda)) \cong$

$$\begin{cases}
\operatorname{ind}_B^G(\mathrm{H}^2(B_1, w \cdot 0 + p\gamma))^{(r-1)} & \text{if } \lambda = p^{r-1}(p\gamma + w.0) \\
 & \quad \text{with } \ell(w) = 2 \text{ or } 0, \\
H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma + p^l w.0, \\
 & \quad \text{with } \ell(w) = 2, \text{and } 0 \leq l < r - 1, \\
H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \\
 & \quad \text{with } 0 < l < r \text{ and } \alpha \in \Delta, \\
H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \\
 & \quad \text{with } 0 \leq l < k < r \text{ and } \alpha, \beta \in \Delta, \\
 & \quad \text{where } k \neq r - 1 \text{ if } \beta = \alpha_2, \\
 & \quad \text{and } k \neq l + 1 \text{ if } \beta = \alpha_1 \\
 & \quad \text{and } \alpha = \alpha_2, \\
H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^{l+1}(\beta + \alpha_1) - p^l\alpha_2, \\
 & \quad \text{with } 0 \leq l < r - 1 \text{ and } \beta \in \Delta, \\
\operatorname{ind}_B^G(N_{G_2} \otimes \gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^{r-1}\alpha_2 - p^l\alpha, \\
 & \quad \text{with } 0 \leq l < r - 1, \text{ and } \alpha \in \Delta, \\
0 & \text{else.}
\end{cases}$$