Multidimensional Inverse Scattering with Fixed-Energy Data

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1 Introduction

In this lecture the author reviews his results on multidimensional inverse scattering. References to the works of other authors can be found in [20]. Five topics are briefly discussed:

- 1) property C with constraints and new type of the uniqueness theorems for inverse scattering,
- 2) inversion of noisy discrete fixed-energy 3D scattering data and error estimates,
- 3) variational principles equivalent to inverse scattering problems,
- 4) low-frequency data inversion,
- 5) asymptotic inverse scattering theory for scattering by small inhomogeneities.

Detailed proofs of the results can be found in the cited references. In this paper the emphasis is on the ideas and formulation of the results.

The inverse problems we consider are: inverse potential, geophysical and obstacle scattering (*IPS*, *IGS*, *IOS*). We recall the statements of these problems. Let

$$[\nabla^2 + k^2 - q(x)] u = 0 \text{ in } \mathbb{R}^3, \quad k = \text{const} > 0 \quad (1.1)$$

$$u = \exp(ik\alpha \cdot x) + A(\alpha', \alpha, k) \frac{\exp(ikr)}{r} + o\left(\frac{1}{r}\right),$$

$$r := |x| \rightarrow \infty, \alpha' := \frac{x}{r}, \alpha \in S^2, \quad (1.2)$$

where S^2 is the unit sphere, α is a given unit vector. $A(\alpha', \alpha, k)$ is the scattering amplitude, and $q(x)$ is the potential. We assume that $q \in Q := \{q : q = \bar{q} \text{ (real-valuedness)}, q(x) = 0 \text{ for } |x| \geq a, q \in L^\infty\}$, $k > 0$ is fixed. The solution to (1.1)-(1.2) is called the scattering solution. We denote B_a the ball centered at the origin with radius a , $u_0 := \exp(ik\alpha \cdot x)$, $M_k := \{z : z \in \mathbb{C}^3, z \cdot z = k^2\}$, $z \cdot y := \sum_{j=1}^3 z_j y_j$, $M_1 := M$, M_k is a non-compact algebraic variety in $\mathbb{C}^3$, $M \cap \mathbb{R}^3 = S^2$.

The IPS consists of finding $q \in Q$, given $A(\alpha', \alpha) := A(\alpha', \alpha, k)$ for all $\alpha', \alpha \in S^2$. We fix $k = 1$ without loss of generality.

The general method for a study of inverse scattering problems, based on property C , was introduced in [4] and in [20] a detailed account of this method is given and its many applications are described. (See [4-42]). Later [29-31,38] the author introduced the notion of *property C with constraints* and found several applications of this notion in inverse scattering.

Let us define *property C* : let L_j be linear partial differential expressions, $j = 1, 2$, $N_j = N_j(D) := \{w : L_j w = 0 \text{ in } D\}$. $D \subset \mathbb{R}^n$, $n \geq 2$, D is a bounded domain; consider subsets of N_j for which the products $w_1 w_2$ are defined, $w_j \in N_j$; let $f \in L^p(D)$, $p \geq 1$, and assume that the equation

$$\int_D f(x) w_1 w_2 dx = 0 \quad \forall w_j \in N_j, \quad (1.3)$$

implies $f(x) = 0$. Then we say that the pair $\{L_1, L_2\}$ has property C .

This definition is rather flexible: we can take $f(x)$ not in $L^p(D)$ but in a different space, including a space of distributions.

The definition of *property C with constraints* is similar, but w_j in (1.3) runs not through all of N_j but through a subset of N_j of finite codimension.

For example, w_j may satisfy finitely many linear constraints:

$$(w_j, h_m) = 0, \quad 1 \leq m \leq M_j < \infty,$$

where (w_j, h_m) is a linear functional on N_j . This functional can be defined by a function or distribution with support in $\overline{D} = D \cup \partial D$.

Let $l_j = \nabla^2 + k^2 - q_j(x)$, $q_j \in Q$.

Theorem 1.1 [29,30,38]. The pair $\{l_1, l_2\}$ has property C with constraints.

Let A_j be the scattering amplitude corresponding to q_j , $j = 1, 2$. From Theorem 1.1 one derives:

Theorem 1.2 [32,33]. If $q \in Q$, $j = 1, 2$, and $A_1(\alpha', \alpha) - A_2(\alpha', \alpha)$ is a finite rank kernel in $L^2(S^2)$, then $q_1 = q_2$.

Sketch of proof: Let $q_1 - q_2 := p(x)$, $A_1 - A_2 := A$. Then one proves [20,p.67] that

$$-4\pi A = \int p(x) u_1(x, \alpha) u_2(x, -\alpha') dx \quad \forall \alpha', \alpha \in S^2. \quad (*)$$

If A is a finite rank kernel, then $A = \sum_{m=1}^M a_m(\alpha) b_m(\alpha')$. Let $(\nu_2(\alpha'), b_m(\alpha')) = 0$, $1 \leq m \leq M$, $\nu_1(\alpha)$ be arbitrary. Multiply $(*)$ by $\nu_1(\alpha) \nu_2(\alpha')$ and integrate over $S^2 \times S^2$ to get

$$0 = \int_{B_a} p(x) w_1(x) w_2(x) dx. \quad (**)$$

$$w_1 := \int_{S^2} u_1(x, \alpha) \nu_1(\alpha) d\alpha, \quad w_2 := \int_{S^2} u_2(x, -\alpha') \nu_2(\alpha') d\alpha'.$$

The set $\{w_1\}$ is dense in $N_1 := \{w : l_1 w = 0 \text{ in } B_a\}$ in $L^2(B_a)$ norm when ν run through all of $L^2(S^2)$, while the set $\{w_2\}$ has codimension $\leq M$ in N_2 (see [33]). By Theorem 1.1, the set $\{w_1 w_2\}$ is total in $L^2(B_a)$. Therefore $(**)$ implies $p(x) = 0$.

Note that the classical uniqueness theorem for ISP [20] says: if $A_1(\alpha', \alpha) = A_2(\alpha', \alpha) \forall \alpha', \alpha \in S^2$ and $q_j \in Q$, then $q_1 = q_2$. Thus, Theorem 2 is a far-reaching generalization of the classical uniqueness theorem. Results in [29,30,38] cover also the case of *IOS* for various boundary conditions, and contain results on inverse spectral problems and inverse boundary value problems. *Roughly speaking, it is proved in [29,30,38] that one can drop any finite number of spectral data or boundary data, and uniqueness of the solutions to inverse spectral problems and inverse boundary value problems is still guaranteed.*

Assume that $q = q(r)$, that is, the potential is spherically symmetric, and that all the phase shifts with $\ell > \ell_0$ vanish, where ℓ_0 is an arbitrary large fixed integer. Then, as follows from Theorem 1.2, $q(r) = 0$.

The *IGS* problem consists of finding an inhomogeneity $v(x)$ in the velocity profile from the surface scattering data. The governing equation is

$$[\nabla^2 + k^2 + k^2 v(x)] U = -\delta(x - y) \text{ in } \mathbb{R}^3, \quad k = \text{const} > 0, \quad (1.4)$$

U satisfies the radiation condition, $v(x) \in Q$, $\text{supp } v \subset \mathbb{R}_-^3$, $\mathbb{R}_-^3 := \{x : x_3 < 0\}$. The surface scattering data are the values $U(x, y) := U(x, y, k) \forall x, \forall y \in P := \{x : x_3 = 0\}$, $k > 0$ is a fixed constant.

The *IOS* problem consists of finding an obstacle D and the boundary condition on $\partial D := S$ from the scattering amplitude $A(\alpha', \alpha)$, $k > 0$ is fixed. The scattering solution is defined by equation (1.1) in $D' := \mathbb{R} \setminus D$ with $q(x) = 0$, equation (1.2), and the boundary condition:

$$u_N + \sigma(s)u = 0 \text{ on } S := \partial D, \quad \sigma(s) \geq 0, \quad (1.5)$$

where u_N is the normal derivative, N is the outer unit normal to S . $\sigma(s) \in C(S)$. It is proved in [41] for Liapunov's boundaries and in [43] for Lipschitz's and even less smooth boundaries (see also [44]) that *IOS* has at most one solution. It is proved in [4,20] that *IGS* has at most one solution. We do not give the strongest results from [20] because of the lack of space. For example, in *IPS* and *IOS* the uniqueness results are proved in [20,41] for the data known $\forall \alpha' \in \tilde{S}_1^2$, and all $\alpha \in \tilde{S}_2^2$ where $\tilde{S}_j^2$ are arbitrary small open subsets of

S^2 , in *IGS* the surface of data can be given on arbitrary small open subsets of P , the assumptions on $q(x)$ and $v(x)$ can be relaxed, etc.

In section 2 we describe an inversion algorithm for *IPS* and give an error estimate for this algorithm. In section 3 we formulate a variational principle equivalent to inverse scattering problems. In section 4 we outline a low-frequency data inversion in *IGS*. In section 5 an asymptotic theory for scattering by a small inhomogeneity is given. In section 6 two old open problems are stated.

2 Inversion of Noisy Fixed-Energy Discrete Scattering Data

Assume that $A_\delta(\alpha', \alpha)$ is known, $\sup_{\alpha', \alpha \in S^2} |A_\delta - A| < \delta$, so that $A_\delta(\alpha', \alpha)$ is the noisy scattering amplitude measured at a fixed energy ($k = 1$) with $\delta > 0$ being the noise level. We do not assume any statistical properties of the noise. At the end, it will be clear that only the values of A_δ at a discrete set of $\alpha', \alpha \in S^2$ are used in the inversion algorithm. We fix an arbitrary $\xi \in \mathbb{R}^3$ and chose θ', θ such that

$$\theta' - \theta = \xi, \quad |\theta| \rightarrow \infty, \quad \theta', \theta \in M. \quad (2.1)$$

This can be done easily and explicitly: choose the coordinates in which $\xi = te_3$, $t > 0$, e_j are unit orthonormal basis vectors, $\theta' = \frac{t}{2}e_3 + y$, $\theta = -\frac{t}{2}e_3 + y$, $y_3 = 0$, $|y| \rightarrow \infty$, $y_1^2 + y_2^2 = y \cdot y = 1 - \frac{t^2}{4}$. There are infinitely many y that satisfy these requirements. Let

$$\tilde{q}(\xi) := \int_{\mathbb{R}^3} \exp(-i\xi \cdot x) q(x) dx.$$

Fix arbitrary a_1 and b such that $a < a_1 < b$. Let $N(\delta) := [\frac{\ln \delta}{\ln |\ln \delta|}]$, where $[x]$ is the closest to x integer. Define

$$A_{\delta\ell}(\alpha) := (A_\delta(\alpha', \alpha), Y_\ell(\alpha'))_{L_2(S^2)},$$

where $Y_\ell(\alpha) := Y_{\ell m}(\alpha)$ are the orthonormal spherical harmonics. Let

$$\hat{A}_\delta(\theta', \alpha) := \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\theta'),$$

where the summation is taken over ℓ and also over m , $-\ell \leq m \leq \ell$, $Y_\ell(\theta)$ is naturally defined for all $\theta \in M$. Let $h_\ell(r)$ be the spherical Hankel function normalized so that $h_\ell(r) \sim \frac{\exp(ir)}{r}$ as $r \rightarrow \infty$, and let

$$\hat{u}_\delta(x, \alpha) := \exp(i\alpha \cdot x) + \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\alpha') h_\ell(r).$$

Define

$$\rho_\delta(\nu) := \exp(-i\theta \cdot x) \int_{S^2} u_\delta(x, \alpha) \nu(\alpha) d\alpha - 1, \quad \theta \in M, \quad \nu \in L^2(S^2),$$

$\mu(\delta) = \exp[-\gamma N(\delta)]$, $\gamma := \ln \frac{a_1}{a} > 0$, $\|\rho\| := \|\rho\|_{L^2(B_b \setminus B_{a_1})}$. Let $c > 0$ be a sufficiently large constant, $\xi \in \mathbb{R}$, and θ', θ satisfy (2.1). Consider the variational problem:

$$|\theta| = \sup, \quad \sup |\theta| := \vartheta(\delta), \quad (2.2)$$

$$\|\rho_\delta(\nu)\| + \|\nu\|_{L^2(S^2)} \exp(|\operatorname{Im} \theta| b) \mu(\delta) \leq c|\theta|^{-1}. \quad (2.3)$$

Lemma 2.1 $\vartheta(\delta) \geq c_0 \frac{|\ln \delta|}{(\ln |\ln \delta|)^2}$, $c_0 = \text{const} > 0$.

Let $\theta(\delta)$, and $\nu_\delta(\alpha)$ be any approximate solution to (2.2) in the following sense: (2.1) and (2.3) hold and $|\theta(\delta)| \geq \frac{1}{2}\vartheta(\delta)$. Define

$$\hat{q} := -4\pi \int_{S^2} \hat{A}_\delta(\theta'(\delta), \alpha) \nu_\delta(\alpha, \theta(\delta)) d\alpha. \quad (2.4)$$

Theorem 2.1 One has

$$|\hat{q} - \bar{q}(\xi)| \leq c_1 \frac{(\ln |\ln \delta|)^2}{|\ln \delta|}, \quad (2.5)$$

where c_1 can be chosen uniformly for $\xi \in \mathbb{R}^3$ and $q \in K \subset Q$, where K is an arbitrary fixed compact subset of Q .

Formula (2.4) gives an inversion algorithm and (2.5) is its error estimate. Since calculation of (2.4) by a cubature formula requires the knowledge of $\hat{A}_\delta(\alpha', \alpha)$ at a discrete set only, formula (2.4) gives inversion procedure for noisy, discrete, fixed-energy, 3-D scattering data.

3 Variational Principles Equivalent to Inverse Scattering Problems

We give here only the variational principle equivalent to *IPS* and refer the reader to [21,42] where similar principles for *IGS* and *IOS* are given.

The basic idea is to use not only the data fitting, but also the basic equation for the scattering solution, namely,

$$u = u_0 - \int_{\mathbb{R}^3} g q u dz, \quad g := \frac{\exp(i|x-y|)}{4\pi|x-y|}, \quad k = 1. \quad (3.1)$$

Consider the variational principle:

$$\begin{aligned} \mathcal{F}(p, f) := & \left\| -4\pi A(\alpha', \alpha) - \int_{B_a} \exp(-i\alpha' \cdot y) f(y, \alpha) dy \right\|_{L^2(S^2 \times S^2)} \\ & + \left\| f(x, \alpha) - p(x) u_0(x, \alpha) + p(x) \int_{B_a} g(x, z) f(z, \alpha) dz \right\|_{L^2(B_a \times S^2)} \\ = & \min \end{aligned} \quad (3.2)$$

Theorem 3.1 *If $A(\alpha', \alpha)$ is the scattering amplitude corresponding to a potential $q \in Q$, then $\min \mathcal{F}(p, f) = 0$, where $\min$ is taken over $p \in L^2(B_q)$ and $f(x, \alpha) \in L^2(B_a \times S^2)$; $\min \mathcal{F}(p, f)$ is attained at the pair $\{q, qu(x, \alpha)\}$ and only at this pair. Here $q \in Q$ is the unique potential corresponding to $A(\alpha', \alpha)$ and $u(x, \alpha)$ is the corresponding scattering solution.*

Remark 3.1 Unfortunately we cannot give any stability results for the solution to (3.2) with noisy data $A_\delta(\alpha', \alpha)$ in place of the exact data $A(\alpha', \alpha)$ in (3.2).

4 Inversion of Low-Frequency Surface Scattering Data

Consider (1.4) and assume that the data $U(x, y, k), \forall x, y \in P, \forall k \in (0, k_0)$ are known, where $k_0 > 0$ is an arbitrary small number. The usual integral equation equivalent to (1.4) (with the radiation condition) is:

$$U = g + k^2 \int_{B_a} g(x, y, k) v(z) U(z, y, k) dz, \quad g := \frac{\exp(ik|k - y|)}{4\pi|x - y|}. \quad (4.1)$$

It is proved in [41] that (4.1) is uniquely solvable for sufficiently small k and the following limit exists:

$$f(x, y) := 16\pi^2 \lim_{k \rightarrow 0} \frac{U - g}{k^2} = \int_{B_a} \frac{v(x) dz}{|x - z||z - y|}, \quad x, y \in P. \quad (4.2)$$

The function $f(x, y)$ is known if $U(x, y)$ is known $\forall x, y \in P$. Equation (4.2) is first kind Fredholm equation for $v(z)$. It is proved in [41] that this equation has at most one solution, this solution is found analytically, and necessary and sufficient condition for a function $f(x, y)$ to be representable by the right-hand side of (4.2) are given [20,21,41]. Numerical experiments in solving (4.2) are reported in [20]. The problem is highly ill-posed, but it was possible to identify numerically the support of $v(x)$ from noisy discrete surface data. A number of other geophysical inverse problems were studied by the above methodology (well-to-well exploration, induction logging, etc, see [20,21,41]). However, in many cases it is desirable to get less information about the inhomogeneity $v(x)$, but to get it in a computationally stable and easy (non-intensive computationally) way. Such a way is discussed in the next section.

5 Asymptotic Inverse Scattering Theory for Small Inhomogeneities

Consider *IGS* problem and assume that

$$\begin{aligned} \text{supp } v &:= D, \quad \text{diam}(D) = d, \\ c_1 d^3 &\leq |D| := \text{meas } D \leq c_2 d^3, \\ \frac{\int_D |v| dx}{|\int_D v dx|} &\leq c, \quad \sup |v| \leq c. \end{aligned} \quad (5.1)$$

Given $U(x, y, k)$ for x and y running through some subsets of P and for k fixed or k running through some finite number of wave numbers, we want to find the location of D and get some idea about its geometry and the intensity $v_D := \int_D v dx$.

In [1-3] more general assumptions are made: ∇^2 is replaced by a second-order elliptic operator. Let $(U - g)k^{-2} := f(x, y)$. We suppressed k -dependence in what follows. Equation (4.1) implies

$$f(x, y) \approx g(x, \tilde{z})g(\tilde{z}, y)v_D, \quad v_D := \int_D v dx \quad (5.2)$$

where $\tilde{z}$ is a point in a neighborhood of D .

Let $\xi_i = (x_i, y_i)$ be a data point, $\hat{g}(\xi_i, z) := g(x_i, z)g(z, y_i)$, $i \in I$, where I is a finite set of indices. By (5.2), $f(\xi)$ is approximately proportional to $\hat{g}(\xi, \tilde{z})$ for some $\tilde{z}$. Let us estimate $\tilde{z}$ by finding global maximum of the correlation coefficient

$$r(z) := \frac{|\sum f(\xi_i)\hat{g}(\xi_i, z)|^2}{\sum |f(\xi_i)|^2 \sum |\hat{g}(\xi_i, z)|^2} = \max, \quad (5.3)$$

where the summation is over $i \in I$. Define

$$z_0 := \int_D z v(z) dz / v_D. \quad (5.4)$$

One can prove [1] the following

Theorem 5.1 Global minimizer $\tilde{z}$ of (5.3) satisfies the following equation.

$$\tilde{z} = z_0 + C(d^{\frac{3}{2}}) \text{ as } d \rightarrow 0 \quad (5.5)$$

A similar result holds when the exact data $U(\xi_i)$ are replaced by the noisy data $U(\xi_i) + n_i$, where n_i are independent random variables with zero mean value and variance σ^2 (see [1]).

If v does not change sign then $\tilde{z}$ belongs to the convex hull of D . If, in addition, D is convex, then $\tilde{z} \in D$. If D is centrally symmetric and connected and $v(x) = v(-x)$ in the coordinate system with the origin at the center of symmetry, then $\tilde{z} = C(d^{\frac{3}{2}})$.

We can now describe a very simple practical algorithm for localization of the small inhomogeneity $v(x)$: calculate $r(z)$ by formula (5.3) and find its global minimizer $\tilde{z}$. The point $\tilde{z}$ determines the position of D with the error $C(d^{\frac{3}{2}})$ as $d \rightarrow 0$. If $\tilde{z}$ is found then an estimate of v_D is calculated by the formula

$$\tilde{v}_D := \frac{\text{Re} \sum f(\xi_i)\hat{g}(\xi_i, \tilde{z})}{\sum |\hat{g}(\xi_i, \tilde{z})|^2}. \quad (5.6)$$

which follows from (5.2). One has

Theorem 5.2.

$$\tilde{v}_d = v_d[1 + C(d^{\frac{3}{2}})] \text{ as } d \rightarrow 0.$$

One can get additional, useful practically, information about $v(z)$ from the surface scattering data, in particular, one can estimate the first and second moments of $v(z)$. (See [1-2] for details.)

A similar approach can be used in *IPS* ([3]). Assuming that

$$d^2(k^2 + \sup |q(x)|) \ll 1 \text{ and } d \sup |q| \ll k,$$

where $d = \text{diam}(\text{supp } q)$, $\text{supp } q := D \subset \mathbb{R}^3$, one can give a computationally easy and stable method for locating D , finding $q_D := \int_D q(x)dx$, the intensity of the potential, and its first and second moments.

Let us sketch the method. We have the well-known equations

$$A(\alpha', \alpha) = -\frac{1}{4\pi} \int_D \exp(-ik\alpha' \cdot x) q(x) u(x, \alpha) dx, \quad (5.7)$$

$$U(x, \alpha) = \exp(ik\alpha \cdot x) - \int_D g(x, y, k) q(y) u(y, \alpha) dy, \quad \alpha, \alpha' \in S^2. \quad (5.8)$$

Thus

$$\begin{aligned} & -4\pi A(\alpha', \alpha) = \\ & \int_D q(z) \exp[ik(\alpha - \alpha') \cdot x] dx - \int_D dx \exp(-ik\alpha \cdot x) q(x) \int_D g(x, y, k) q(y) dy. \end{aligned} \quad (5.9)$$

Let

$$x_0 := \int_D x q(x) dx / q_D, \quad \kappa := k^2 + Q, \quad Q := \sup_{x \in \mathbb{R}^3} |q|. \quad (5.10)$$

One can prove that (5.9) implies

$$-4\pi A(\alpha', \alpha) = \exp[ik(\alpha - \alpha') \cdot x_0] q_D [1 + C(\kappa d^2)], \quad d \rightarrow 0. \quad (5.11)$$

Thus

$$4\pi |A(\alpha', \alpha)| = |q_D| (1 + \kappa d^2); \quad 0 \leq \arg[-A(\alpha', \alpha)] := c(\alpha', \alpha) < 2\pi, \quad (5.12)$$

$$c(\alpha', \alpha) = k(\alpha - \alpha') \cdot x_0 + 2\pi n(\alpha', \alpha) + C(\kappa d^2) \quad (5.13)$$

where $n(\alpha', \alpha)$ is an integer.

Let $\beta := \alpha - \alpha'$. If $\alpha, \alpha' \in S^2$ then $|\beta| \leq 2$. If β is given, $\beta \in \mathbb{R}^3$, $|\beta| \leq 2$, and $\mathcal{P}$, a plane containing the origin, is fixed, then one can find uniquely $\alpha, \alpha' \in S^2 \cap \mathcal{P}$, such that $\beta = \alpha' - \alpha$. Therefore (5.13) can be written as

$$c(\beta, \mathcal{P}) = k\beta \cdot x_0 + 2\pi n(\beta, \mathcal{P}) + C(\kappa d^2), \quad |\beta| \leq 2. \quad (5.14)$$

From the second equation (5.12) it follows that $c(\beta, \mathcal{P})$ is discontinuous only for those β for which $c(\beta, \mathcal{P}) = 0$ or $c(\beta, \mathcal{P}) = 2\pi - 0$. Suppose the origin is inside $B_a \supset D := \text{supp } q(x)$ and assume that $x_0 \in B_a$. This will happen, for example, if the ratio $\frac{\int_{B_a} |q| dx}{|\int_{B_a} q dx|}$ is not too large. For instance, if $q(x)$ does not

change sign, then this ratio is 1 and $x_0 \in B_a$. Define $\Delta := \pi(ka)^{-1}$ and let $|\beta_2 - \beta_1| < \Delta$. Then $|k(\beta_2 - \beta_1) \cdot x_0| \leq ka|\beta_2 - \beta_1| < \pi$. Fix any $\beta_1 \neq \beta_2$ such that $|\beta_1 - \beta_2| < \Delta$. Consider the following three possibilities:

- 1) $-\pi + C(\kappa d^2) < c_2 - c_1 < \pi - C(\kappa d^2)$, where $c_j = c(\beta_j, \mathcal{P}_j)$; then $(\beta_2 - \beta_1) \cdot x_0 = k^{-1}(c_2 - c_1) + C(\kappa k^{-1} d^2)$;
- 2) $c_2 - c_1 > \pi + C(\kappa d^2)$, then $n_2 = n_1 + 1$, $n_j := (\beta_j, \mathcal{P}_j)$; thus $(\beta_2 - \beta_1)x_0 = k^{-1}[c_2 - c_1 - 2\pi + C(\kappa d^2)]$;
- 3) $c_2 - c_1 < -\pi - C(\kappa d^2)$, then $n_2 - n_1 = -1$, and $(\beta_2 - \beta_1)x_0 = k^{-1}[c_1 - c_1 + 2\pi + C(\kappa d^2)]$.

Let us estimate the point x_0 which localizes the support of the potential.

Let $\{\beta_j, c_j\}$ be the data. For all pairs j_1, j_2 such that $|\beta_{j_1} - \beta_{j_2}| < \Delta$, $\beta_{j_1} \neq \beta_{j_2}$, consider the equations

$$x \cdot b_l = k^{-1} p_l, \quad 1 \leq l \leq L, \quad b_l := \beta_{j_2} - \beta_{j_1}, \quad p_l := c_{j_2} - c_{j_1} + 2\pi s. \quad (5.15)$$

Index l numerates the pairs $\{j_1, j_2\}$, $s = 0, 1$, or -1 according to which of the possibilities 1), 2) or 3) occurs.

Solve (5.15) for x by solving the system

$$B^t B \tilde{x} = k^{-1} B^t p, \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_L \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \vdots & \vdots & \vdots \\ b_{L1} & b_{L2} & b_{L3} \end{pmatrix} \quad (5.16)$$

If the vectors $\frac{\beta}{|\beta|}$, $\beta := \alpha', \alpha$, are distributed uniformly on S^2 and if there are many pairs $\{\beta_{j_1}, \beta_{j_2}\}$, $\beta_{j_1} \neq \beta_{j_2}$, such that $|\beta_{j_1} - \beta_{j_2}| < \Delta$, then one can find $\tilde{x}$ from (5.16).

Theorem 5.3 One has $|\tilde{x} - x_0| = C(kd^2 + k^{-1}Qd^2)$ as $d \rightarrow 0$.

If $\tilde{x}$ is found, then one estimates the intensity of the potential, q_D , by the formula

$$\tilde{q}_D = q_D[1 + C(d^2 k^2 + Qd^2)] \text{ as } d \rightarrow 0. \quad (5.17)$$

As above, one can find the first and second moments of $q(x)$ from the scattering data.

6 Open Problems

We mention just two major problems each of which has been open for about half a century:

- 1) *Do the data $A(\alpha', \alpha_0, k)$ (or $A(-\alpha, \alpha, k)$) determine $q(x)$ uniquely?* Here $\alpha_0 \in S^2$ is fixed, α', α run through all of S^2 and k runs through $(0, \infty)$.
- 2) *Do the data $A(\alpha', \alpha_0, k_0)$ determine D uniquely?*

Here $\alpha_0 \in S^2$ and $k_0 > 0$ are fixed, α' runs through an open subset of S^2 , and $D \subset \mathbb{R}^3$ is a bounded domain with a Lipschitz (or smooth) boundary on which the Dirichlet (or Neumann, or Robin) boundary condition is assumed.

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