Name:

MATH 512 Intro to Modern Algebra – **Exam I**

Wednesday, September 22, 2004

Check that that you have all four pages - note that the pages are double-sided

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1. (18 points) Define what it means for $< G, * >$ to be a group.

(b) Prove that $< \mathbb{R}^*, * >$ is a group with $a * b = \frac{1}{2}ab$.

PG SCORE/18

2. (26 points) (a) Let $< S, * >$ and $< S', *' >$ be binary algebraic structures. What properties must the function $\phi : S \to S'$ satisfy in order to be an isomorphism?

(b) Prove that $\phi(x) = x^3$ is an isomorphism from $< \mathbb{R}^*, \cdot >$ to $< \mathbb{R}^*, \cdot >$.

(c) Prove that if $\phi : S \to S'$ is an isomorphism for $< S, * >$ and $< S', *' >$ and e is an identity for S then $\phi(e)$ is an identity for S'.

(d) For each of the following pairs decide whether they are isomorphic. If so give (without proof) an isomorphism from the first to the second. If not give a reason why there can be no such isomorphism.

(i) $< \mathbb{R}, + >$ and $< \mathbb{R}^+, \cdot >$

(ii) $< \mathbb{Q}, + >$ and $< \mathbb{R}, + >$

PG SCORE/26

3. (14 points)

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \; : \; a, b \in \mathbb{R}, \; a \neq 0 \right\}$$

(a) Show that the set H is closed under matrix multiplication.

(b) Show that H contains an identity under multiplication.

(c) Show the additional property needed to prove that $< H, \cdot >$ is a subgroup of the group $< GL(2, \mathbb{R}), \cdot >$.

(d) Is $< H, \cdot >$ abelian?

4. (12 points) Suppose that $G = \{e, a, b, c, d, f\}$ is a group with identity e.

(a) Complete the group table.

$*$	e	a	b	c	d	f
e	e			c	d	f
a	a	b	e	f		
b	b	e	a			
c	c			e		b
d	d	f			e	
f	f	c	d	a		

(b) Is G abelian?

(c) G must be isomorphic to either $\mathbb{Z}_6$ or the dihedral group D_3 (symmetries of the triangle). Which is it?

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5. (6 points) Prove that if $< G, * >$ is a group then for all a, b in G

$$(a * b)^{-1} = b^{-1} * a^{-1}.$$

Indicate clearly which group properties you are using at each step.

6. (24 points) Circle True (T) or False (F).

T F (a) $< \mathbb{Z}, + >$ is a group.

T F (b) $< \mathbb{R}, \cdot >$ is a group.

T F (c) $< \mathbb{Q}^+, \cdot >$ is a group.

T F (d) The set $H = \left\{ \begin{bmatrix} a & -a \\ 0 & 0 \end{bmatrix} \ : \ a \in \mathbb{Q}^* \right\}$ has no identity under multiplication.

T F (e) $\phi(f) = f'$, the derivative of f, is an isomorphism from $< \mathbb{Q}[x], + >$ to $< \mathbb{Q}[x], + >$.

T F (f) If $e \in S$ satisfies $a * e = a$ for all a in S then e is the identity for S.

T F (g) A group G is abelian if $(a * b)^{-1} = a^{-1} * b^{-1}$ for all a, b in G.

T F (h) If $a, b \in G$ and G is not abelian then $a * b \neq b * a$.

T F (i) Suppose that $a \sim b$ if $a + b$ is even, then $\sim$ is an equivalence relation on $\mathbb{Z}$.

T F (j) The relation $a * b = a^2 b^2$ on $\mathbb{Z}$ is associative.

T F (k) The relation $a * b = a^2 b^2$ on $\mathbb{Z}$ is commutative.

T F (l) $< \mathbb{R}^*, \cdot >$ and $< \mathbb{R}, + >$ are isomorphic.

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