

# Spring 2003 Math 510 Final Exam

Time: 4:10pm–6:00am, Friday, May 16 2003

Name: \_\_\_\_\_

Please read the problems carefully and do **all** you are asked to do. You need to show your work to indicate how you do each problem. Your proofs and explanation should be literally clear.

Total/100	#1/10	#2/10	#3/10	#4/10	#5/10	#6/10	#7/10	#8/10	#9/10	#10/10

1. (10pts) There is a pile of unsorted baseball cards of players  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$ . The owner plans to make them into small packages of fixed number of cards so that each package contains either at least 5 cards of player  $A$ , or at least 6 cards of player  $B$ , or at least 7 cards of player  $C$ , or at least 8 cards of player  $D$ , or at least 9 cards of player  $E$ . Help the owner to determine the minimal number of cards in each package so that if the packages are packed by a machine randomly, the above requirement is always guaranteed. Make sure to convince the owner why your suggestion works. (What principle do you use?).

2. (10pts) There are 50 indistinguishable apples, 3 bananas, and one orange to be distributed among 5 children such that each child gets at least one fruit but no one gets more than one banana and no one gets both orange and banana. Compute the number of ways to distribute the fruits.

3. (10pts) In how many ways can ten gentlemen, six ladies, and a dog be seated at a round table if no two ladies are to sit next to each other?

4. (10pts) By integrating a binomial expansion, compute that number (for each fixed integer  $n > 1$ )

$$1 + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \cdots + \frac{1}{n+1} \binom{n}{n}.$$

5. (10pts) How many integers are there from 0 to 999,999 (inclusive) having each of 2, 5, and 8 among their digits? (It is possible for a digit to appear more than once.) (*Hint*: How many numbers in the range that does not have 2 as a digit.)

6. (10pts) Determine the number of ways to place 6 non-attacking rooks on the  $6 \times 6$ -board with forbidden positions as shown.

X	X				
X	X				
		X	X		
		X	X		
				X	X
				X	X

7. (10pts). For the given sets  $A_1 = \{a, b, c\}$ ,  $A_2 = \{a, b, c, d, e, f\}$ ,  $A_3 = \{a, b\}$ ,  $A_4 = \{b, c\}$ ,  $A_5 = \{a, c\}$ ,  $A_6 = \{a, c, f\}$ , determine whether the family  $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$  has a SDR (system of distinguished representatives). If not, draw the bipartite graph and then determine a max-matching. You need to explain why your answer is maximal.

8. (10pts) Use the deferred acceptance algorithm to find the men-optimal stable complete marriages for the preferential matrix (with A,B,C, D being women)

$$\begin{array}{l} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 3,1 & 4,1 & 2,2 & 1,4 \\ 3,2 & 1,2 & 4,1 & 2,3 \\ 2,3 & 3,4 & 4,3 & 1,2 \\ 3,4 & 2,3 & 1,4 & 4,1 \end{bmatrix}$$

9. (10pts) Determine all trees that have open Hamilton chains. Explain why they are *all*?

10 (10pts) For the recurrence relation  $h_n = h_{n-1} + 6h_{n-2}$ ,  $h_0 = 2$ , and  $h_1 = 6$ ,

(a) Compute the generating function  $g(x)$  for the sequence  $h_0, h_1, h_2, h_3, \dots$

(b) Use the generating function you found in (a) to compute the solution for  $h_n$ .