Name:

MATH 506 Number Theory – Exam III

Monday April 28, 2003

Check that that you have all three pages - note that page two is on the back of page one

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Recall the multiplicative functions

$$\begin{aligned} \mu(n) &= \text{ the Möbius function,} \\ \phi(n) &= \text{ the Euler phi-function,} \\ \sigma(n) &= \text{ the sum of the positive divisors of } n. \end{aligned}$$

You can assume multiplicativity when appropriate.

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1. (12 points) Use the Chinese Remainder Theorem to solve the system of simultaneous congruences:

$$\begin{aligned} x &\equiv 1 \pmod 5, \\ x &\equiv 2 \pmod 8, \\ x &\equiv 3 \pmod 9. \end{aligned}$$

2. (12 points) (a) Find $(126, 255)$ and write this g.c.d. as a linear combination of the two numbers.

(b) Find all the solutions **mod 255** of the following congruence or say why no solutions exist:

$$126x \equiv 15 \pmod{255}.$$

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3. (13 points) (a) If x, y, z is a primitive Pythagorean triple $x^2 + y^2 = z^2$, with x even, then

$$x = \_\_\_\_\_\_\_\_, \qquad y = \_\_\_\_\_\_\_\_\_\_\_\_, \qquad z = \_\_\_\_\_\_\_\_\_\_\_\_,$$

for some coprime positive integers u, v, not both odd, with $u > v$.

(b) Find all the right-angled triangles with coprime integer sides and base length $x = 20$.

(c) Find all the right-angled triangles with coprime integer sides and base length $y = 13$.

4. (22 points) Circle True (T) or False (F).

T F (a) The equation $3x \equiv 2 \pmod 9$ has no solution.

T F (b) There are no integers n with $\phi(n) = 13$.

T F (c) By Wilson's Theorem $13! \equiv -1 \pmod{12}$.

T F (d) If n and m are coprime positive integers with nm a cube, then n and m are both cubes.

T F (e) If $f(n)$ is multiplicative then $f(n)^2$ is multiplicative.

T F (f) The number $n = 2^{13}(2^{14} - 1)$ is perfect.

T F (g) There is a primitive Pythagorean triple $x^2 + y^2 = z^2$ with $x = 6$.

T F (h) The set $\{1, -1\}$ is a reduced residue system mod 6.

T F (i) The set $\{-2, 1, 3, 7, 10\}$ is a complete residue system mod 5.

T F (j) If $x \equiv 5 \pmod 6$ then $x \equiv 2 \pmod 3$.

T F (k) $\mu(27) = -1$.

5. (12 points) (a) Evaluate $\phi(300) =$

(b) Find 3 values of n with $\phi(n) = 4$. (Bonus points if you find more!)

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6. (10 points) Define

$$F(n) = \sum_{d|n} \mu(d)\sigma(d).$$

(a) Give a formula for the value at prime powers $F(p^k)$.

(b) Evaluate $F(300) =$

7. (10 points) Suppose that $f(n)$ is a function satisfying

$$n^3 = \sum_{d|n} f(d).$$

(a) Use the Möbius Inversion Formula to write an expression for $f(n)$.

(b) Evaluate $f(8) =$

8. (9 points) The number $2620 = 2^2 \cdot 5 \cdot 131$ is one member of an amicable pair.

(a) Evaluate $\sigma(2620) =$

(b) Is 2620 abundant, perfect or deficient?

(c) What is the other member of the amicable pair?

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