Advanced Calc. II FINAL EXAM Spring 88

Name.

Instructions: Do one problem from each of the following 8 sets of problems. Each problem is worth 25 pts. You may do additional problems for 5 pts each. Start the additional problems on a separate sheet and label them "Additional".

1 A] Suppose that f is a function of bounded variation on $[0,1]$ with variation V_f and that C is the curve given by $\vec{\gamma}(t) = (t, f(t))$, $0 \le t \le 1$. Prove that C is rectifiable and show that its length L_C satisfies $L_C \le 1 + V_f$. (use definition of rectifiable).

B] i) Show that $g(t) = t^3 \sin\left(\frac{1}{t}\right)$ is continuously differentiable on the interval $[0,1]$.

ii) Prove that the curve C given by $\vec{\gamma}(t) = \left(t, t^3 \sin\left(\frac{1}{t}\right)\right)$ $0 \le t \le 1$ is rectifiable and express its arc length as a definite integral. (use (i) and an appropriate Theorem).

2 A] Give an ε/δ proof of $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y}{|x|+|y|} \sin(xy) = 0$.

B] Let D be a compact set and A a fixed point not in D. Define $f: D \to \mathbb{R}$ by $f(\vec{P}) = |\vec{P} - \vec{A}|$.

i) Give an ε/δ proof to show f is continuous on D.

ii) Why does f attain a minimum value on D. What is the geometric ~~significance~~ interpretation of this minimum value?

$\boxed{3}$ A] Suppose that $\vec{F}: S \to \mathbb{R}^m$ is a continuous function defined on the arc-connected subset S of $\mathbb{R}^n$. Prove that $\vec{F}(S)$ is arc-connected.

B] Suppose that $\{\vec{P}_n\}$ is a Cauchy sequence of points in a closed set S (in $\mathbb{R}^n$) and that $f: S \to \mathbb{R}$ is a continuous function on S. Prove that $\{f(\vec{P}_n)\}$ is a Cauchy sequence of real numbers.

$\boxed{4}$ A] Suppose that $\vec{F}: D \to \mathbb{R}^m$ is differentiable at a point $\vec{p} \in D$ with derivative $\mathcal{J}_{\vec{p}}$. Prove that for any unit vector $\vec{\alpha}$, $\nabla_{\vec{\alpha}} \vec{F}(\vec{p}) = \mathcal{J}_{\vec{p}}(\vec{\alpha})$. (use definitions).

B] Let f be a real valued function defined on $\mathbb{R}^n$ such that

$$f(t\vec{P}) = t^k f(\vec{P}) \quad \forall \vec{P} \in \mathbb{R}^n \text{ and } \forall t \in \mathbb{R},$$

where k is a fixed positive integer.

i) Show that $f(\vec{0}) = 0$.

ii) Let $\vec{\alpha}$ be any unit vector in $\mathbb{R}^n$. Compute $\nabla_{\vec{\alpha}} f(\vec{0})$. (Your answer will depend on k).

5\] A\] Let

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Prove that f is differentiable on $\mathbb{R}^2$. (All of $\mathbb{R}^2$).

B\] Let D be an open connected set in $\mathbb{R}^n$ and $f: D \to \mathbb{R}$ be a differentiable function on D with $f'(\vec{P}) = 0$ (the zero transformation) $\forall\, \vec{P} \in D$. Prove that f is constant on D. (You may assume that any two points in D can be connected by a polygonal line segment)

6\] A\] Verify that $(1,1,1)$ is a critical point of the function $f(x,y,z) = -\frac{1}{4}\left(x^{-4} + y^{-4} + z^{-4}\right) + yz - x - 2y - 2z$ and classify it as a local max, min or saddle pt.

B\] Using a Taylor expansion about $(0,0)$ we can write $\sin(x^2+y) = L(x,y) + E(x,y)$ where $L(x,y)$ is the first order approximation and $E(x,y)$ is the error.

i) Find $L(x,y)$ and $E(x,y)$, explicitly, expressing $E(x,y)$ in terms of a point $\vec{c}$. (Where is $\vec{c}$ located?).

ii) Find an upper bound for $|E(x,y)|$ for $|x| \le 1$, $|y| \le 1$.

7 A] Let f, g be continuously differentiable functions on $\mathbb{R}^3$ and C be the curve defined by the intersection of the surfaces $f(x,y,z) = 0$ and $g(x,y,z) = 0$. Assume that $(0,0,0) \in C$, that is, $f(\vec{0}) = g(\vec{0}) = 0$.

i) State sufficient conditions in order to be able to solve for y and z in terms of x on a NBD of $(0,0,0)$. (That is, to give a parametric eqns for C in terms of x).

ii) Give a formula for the tangent vector to C at $(0,0,0)$ in terms of $f_x, f_y, \ldots, g_z$ each evaluated at $(0,0,0)$.

B] Consider the equation $3x + x^5 + yx + y^5 + y = 0$

i) Why can we solve for y as a continuously differentiable function $y = f(x)$ on a NBD of $(0,0)$?

ii) Find the first order approximation of $f(x)$ expanded about 0.

8 A) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ such that $f_n \to f$ uniformly on $[0,1]$. Prove that f is continuous on $[0,1]$.

B] Consider the sequence of functions $\left\{ \frac{n^2 x}{1 + n^3 x^2} \right\}$.

i) Does this sequence converge uniformly on $[0,1]$?

ii) Prove that the sequence converges to a continuous function on the interval $[1, \infty)$.