Advanced Calculus Test 2

Do any 5 problems, 20 pts each.

1] Suppose that $\vec{F}, \vec{G} : D \to \mathbb{R}^m$ are differentiable at a point $\vec{P} \in D$. Prove that $\vec{F} + \vec{G}$ is differentiable at $\vec{P}$ and that $(\vec{F} + \vec{G})'(\vec{P}) = \vec{F}'(\vec{P}) + \vec{G}'(\vec{P})$. (use definition of diff.)

2] Let $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & ; (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Compute $\frac{\partial f}{\partial x}(\vec{0})$, $\frac{\partial f}{\partial y}(\vec{0})$ and $\nabla_{\vec{\alpha}} f(\vec{0})$ where $\vec{\alpha} = \frac{1}{\sqrt{2}}(1,1)$.

3] Let $\vec{F}(x,y) = (xy, x-y, x^2)$ and $\vec{G}$ be a function differentiable at $(0,-1,0)$ with $J_{\vec{G},(0,-1,0)} = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 1 \end{bmatrix}$

Set $\vec{H} = \vec{G} \circ \vec{F}$.

a) Why is $\vec{F}$ differentiable at $(0,1)$?

b) Why is $\vec{H}$ diff. at $(0,1)$?

c) Find $J_{\vec{H},(0,1)}$

4] Classify the critical points of

$$f(x,y,z) = x^2 + xz - 3\cos y + z^2$$

5] Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be a <u>nonsingular</u> linear transformation. Prove that $\exists$ a $\delta > 0$ such that $|L(\vec{u})| \geq \delta |\vec{u}|$, $\forall \vec{u} \neq \vec{0} \in \mathbb{R}^n$.
(Hint: First consider the sphere of radius one $\{|\vec{u}| = 1\}$.)

6] For any differentiable vector function $\vec{F} = (f_1, f_2, f_3)$ we define Curl $\vec{F}$ by

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

Suppose that $f(x,y,z)$ has continuous 2^{nd} partial derivatives on an open set D. Prove that Curl $\nabla f \equiv \vec{0}$ on D.

7] State and prove the Mean Value Theorem for a function $f(x_1, x_2, \ldots, x_n)$ differentiable on a NBD N in $\mathbb{R}^n$.