Solutions to Test 1

3] It suffices to find λ, β so that $\left[\vec{w} - (\lambda\vec{u} + \beta\vec{v})\right]\cdot\vec{u} = 0$ and $\left[\vec{w} - (\lambda\vec{u} + \beta\vec{v})\right]\cdot\vec{v} = 0$ $\Leftrightarrow$ $\vec{w}\cdot\vec{u} - \lambda\,\vec{u}\cdot\vec{u} - \beta\,\vec{v}\cdot\vec{u} = 0$ and $\vec{w}\cdot\vec{v} - \lambda\vec{u}\cdot\vec{v} - \beta\,\vec{v}\cdot\vec{v} = 0$ $\Leftrightarrow$ $\vec{w}\cdot\vec{u} - \lambda = 0$ and $\vec{w}\cdot\vec{v} - \beta = 0$ (since $\vec{u} \perp \vec{v}$, $|\vec{u}| = |\vec{v}| = 1$)

$$\Leftrightarrow \lambda = \vec{w}\cdot\vec{u}\ ,\ \beta = \vec{w}\cdot\vec{v}$$

4] Notes & Book

5] Notes & Book

6] Notes

7] Let $Q \in S^c$. For each $P \in S$ let $B(P, \delta_P)$ be the open ball of radius $\delta_P = \frac{1}{2}|\vec{P} - \vec{Q}| > 0$. Let $\mathcal{U} = \{B(P, \delta_P) : P \in S\}$ ~~be~~ a covering of S by open sets. Since S is compact $\exists$ finite subcovering $\{B(P_1, \delta_{P_1}), \ldots, B(P_k, \delta_{P_k})\}$. Let $\delta = \min\limits_{i=1,\ldots,k} \delta_{P_i}$. We claim that the open ball $B(Q, \delta)$ is contained in S^c, and therefore S^c is open. Suppose $\vec{P} \in B(Q, \delta) \cap B(P_i, \delta_{P_i})$ for some $i \le k$. Then $|Q - P_i| \le |Q - P| + |P - P_i| < \delta + \delta_{P_i} \le 2\delta_{P_i} = |P_i - Q|$, a contradiction. Therefore $B(Q, \delta) \cap B(P_i, \delta_{P_i}) = \emptyset$ $\forall i$ and so $B(Q, \delta) \cap S = \emptyset$. Since S^c is open, S is closed.

8] First we show $\{\vec{P}_n\}$ is Cauchy. Note, for $n > m$

$$|\vec{P}_n - \vec{P}_m| \le |\vec{P}_n - \vec{P}_{n-1}| + |\vec{P}_{n-1} + \vec{P}_{n-2}| + \cdots + |\vec{P}_{m+1} - \vec{P}_m|$$
$$\le \frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^{m+1}} = \frac{1}{2^{m+1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}\right) < \frac{1}{2^m}$$

Thus if N is such that $\frac{1}{2^N} < \varepsilon$, then $|\vec{P}_n - \vec{P}_m| < \frac{1}{2^m} < \frac{1}{2^N} < \varepsilon$ for $n > m > N$. $\therefore$ $\{\vec{P}_n\}$ converges. Since S is closed $\{\vec{P}_n\}$ converges to a point in S (Corollary 8.2b).

9] First we observe that for any $u \in \mathbb{R}^3$,

$$u \otimes u = -u \otimes u \quad (\text{by } (i)) \quad \text{and so} \quad u \otimes u = 0.$$

Let $xi + yj + zk,\ ui + vj + wk \in \mathbb{R}^3$. Then, by linearity,

$$(xi+yj+zk)\otimes(ui+vj+wk) = xu\, i\otimes i + yu\, j\otimes i + zu\, k\otimes i + yv\, j\otimes j + xv\, i\otimes j + zv\, k\otimes j + xw\, i\otimes k + yw\, j\otimes k + zw\, k\otimes k$$

$$= (xv - yu)\, i\otimes j + (yw - zv)\, j\otimes k + (zu - xw)\, k\otimes i$$

$$= (xv - yu)\, k + (yw - zv)\, i + (zu - xw)\, j$$

$$= (yw - zv,\ zu - xw,\ xv - yu)$$

$$= \begin{vmatrix} i & j & k \\ x & y & z \\ u & v & w \end{vmatrix} = (x, y, z) \times (u, v, w)$$