Solutions to HW 7:

HW 7 1) Let $\varepsilon > 0$ be given. Since $a_n \to L$, $\exists N \ni n > N \Rightarrow |a_n - L| < \varepsilon$.
Now, since n_k is an increasing sequence $n_k \geq k$ $\forall k \in \mathbb{N}$. Set $K = N$. Then for $k > K$, $n_k \geq k > K = N$ and so $|a_{n_k} - L| < \varepsilon$. QED.

2) If $\{a_n\}$ converges to L then so does every subsequence of $\{a_n\}$, by problem 1. Therefore L is the only cluster point.

b) If $a_n \not\to L$ as $n \to \infty$, then $\exists \varepsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists n > N$ with
$\star$ $|a_n - L| > \varepsilon$. Let $\{a_{n_k}\}$ be the subsequence consisting of all terms of the original sequence satisfying $\star$. Since $\{a_n\}$ is bounded, so is $\{a_{n_k}\}$ and so by Bolzano-Weierstrass, $\{a_{n_k}\}$ has a cluster pt C. Clearly $C \neq L$ by $\star$. Then $\{a_{n_k}\}$ has a subsequence converging to C, but this subsequence is also a subsequence of $\{a_n\}$, and so C is a cluster pt of $\{a_n\}$, a contradiction.

3] Since $\{T_n\}$ converges it is Cauchy so given $\varepsilon > 0$ $\exists N \ni n, m > N \Rightarrow |T_n - T_m| < \varepsilon$.
Now $|S_n - S_m| = \left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k| = T_n - T_m = |T_n - T_m|$
for $n > m$
Thus if $n > m > N$ then $|S_n - S_m| \leq |T_n - T_m| < \varepsilon$, so $\{S_n\}$ is Cauchy.
Therefore, $\{S_n\}$ converges. If a series converges absolutely, then it converges.

4] a) Since f is unif cont on (a,b), given $\varepsilon > 0$ $\exists \delta > 0 \ni$
$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Since $\{a_n\}$ converges, it is Cauchy, so ~~gave~~
$\exists N \ni n, m > N \Rightarrow |a_n - a_m| < \delta$. We assume $a_n \in (a,b)$ $\forall n \in \mathbb{N}$.
Then for $n, m > N$ we have (letting $x = a_n$, $y = a_m$) $|f(a_n) - f(a_m)| < \varepsilon$.
$\therefore$ $\{f(a_n)\}$ is Cauchy, and so it converges

4b) Let $\{a_n\}$ be sequence with $a_n < b$ and $a_n \to b$ as $n \to \infty$. By (a) $\{f(a_n)\}$ converges. Say $f(a_n) \to L$ as $n \to \infty$. Then given $\varepsilon > 0$ $\exists N_1 \ni$ $n > N_1 \Rightarrow |f(a_n) - L| < \varepsilon/2$. Since f is unif. cont., $\exists \delta > 0 \ni$

$\star$ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2$. Finally, since $a_n \to b$ $\exists N_2 \ni$ $n > N_2 \Rightarrow |a_n - b| < \delta$. Suppose $n > \max(N_1, N_2)$, ~~then since~~ and that $b-\delta < x < b$. Then since $a_n, x \in (b-\delta, b)$ we have $|a_n - x| < \delta$ and so by $\star$, $|f(a_n) - f(x)| < \varepsilon/2$. Thus

$$|f(x) - L| = |f(x) - f(a_n) + f(a_n) - L| \le |f(x) - f(a_n)| + |f(a_n) - L|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad , \text{ for } b-\delta < x < b \quad . \qquad QED.$$

5 a) Let $x \in (a,b)$. Set $\delta = \min(x-a, b-x)$. Then $(x-\delta, x+\delta)$ is a NBD of x with $(x-\delta, x+\delta) \subset (a,b)$. $\therefore$ (a,b) is open.

b) Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Gamma}$ be a covering of $[a,b]$ by open sets. For any $x \in [a,b]$, $\exists U_{\alpha_x} \in \mathcal{U}$ with $x \in U_{\alpha_x}$, since $\mathcal{U}$ covers $[a,b]$. Since U_{α_x} is open, $\exists$ open interval I_x with $x \in I_x \subset U_{\alpha_x}$. Then $\mathcal{I} = \{I_x\}_{x \in [a,b]}$ is a covering of $[a,b]$ by open intervals. By Heine-Borel $\mathcal{I}$ has a finite subcovering $\{I_{x_1}, I_{x_2}, \ldots, I_{x_k}\}$. Then $\{U_{\alpha_{x_1}}, U_{\alpha_{x_2}}, \ldots\ldots, U_{\alpha_{x_k}}\}$ is a finite subcovering of $\mathcal{U}$, since

$$[a,b] \subset \bigcup_{i=1}^{k} I_{x_i} \subset \bigcup_{i=1}^{k} U_{\alpha_{x_i}} .$$