1. Let $f: S \to T$ (where S and T are subsets of $\mathbb{R}$) be a function with domain S and range T. Prove that f has an inverse $g: T \to S$ iff f is one-to-one on S. (Verify all 3 properties of inverse function for one direction).

Pf: Suppose $f: S \to T$ and f has an inverse $g: T \to S$.
Suppose $x_1, x_2 \in S$ with $f(x_1) = f(x_2)$. Then $g \circ f(x_1) = g \circ f(x_2)$
$\Rightarrow x_1 = x_2$ $\therefore$ f is one-to-one on S.

Now Prove the converse:
Suppose $f: S \to T$ is one-to-one function. Define $g: T \to S$ by setting $g(y) = x$ where x is the unique pt in S such that $f(x) = y$. Then $g \circ f(x) = g(y) = x$ $\forall x \in S$
and $f \circ g(y) = f(x) = y$ $\forall y \in T$ $\Rightarrow$ g is an inverse of f

2. Let f be a real valued function defined on $I = (a, b]$. Suppose that for any monotone increasing sequence $\{a_n\}$ with $a_n \to b$ as $n \to \infty$ we have $f(a_n) \to f(b)$. Prove, using $\varepsilon - \delta$, that f is left continuous at b. (Hint: proof by contradiction as for Thm 2.4(i))

Pf: Proof by contradiction: Suppose that $\lim_{x \to b^-} f(x) \neq f(b)$
Then $\exists\, \varepsilon > 0 \ni \forall \delta > 0$ $\exists\, x \in (b-\delta, b) \ni |f(x) - f(b)| \geq \varepsilon$.
Now Define $\{a_n\}$ inductively as follows: for $\delta = \min(\frac{1}{n}, b - a_{n-1})$, $n \in \mathbb{N}$
$\exists\, a_n \ni b - \delta < a_n < b$ and $|f(a_n) - f(b)| \geq \varepsilon$
However, $b - \frac{1}{n} < a_n < b$, then as $n \to \infty$ $a_n \to b$ by the squeeze law.
Also, $b - a_n < b - a_{n-1} \Rightarrow a_n > a_{n-1} \Rightarrow \{a_n\}$ is a monotone increasing sequence and converges to b. It follows that $f(a_n) \to f(b)$, a contradiction
$\therefore \lim_{x \to b^-} f(x) = f(b) \Rightarrow f$ is left continuous at b

3. Let I be an interval of real numbers. A subset S of I is called a <u>dense subset</u> of I if every point of I is a <u>limit point</u> of S.

(a) Prove that S is a dense subset of I iff every open subinterval J of I contains at least <u>one</u> point of S. (In particular, the set of rationals in I and the set of irrationals in I are both dense subsets of I.)

pf: $\Rightarrow$

Suppose S is a dense subset of I and let J be an open subinterval of I. Since every point of I is a limit point of S, every pt. of J is a limit pt. of S. Let a be the mid-point of J and let $\delta = \frac{|J|}{2}$. Then the δ-NBD of a, i.e, the interval J, contains a point of S

$\Leftarrow$

Suppose every open subinterval of I contains at least one point of S. Prove that S is a dense subset of I. i.e. prove that every point of I is a limit point of S

Let a be any point in I. Then $\forall \delta > 0$, the interval $(a-\delta, a+\delta)$ will include some subinterval J of I with $a \notin J$. $\Rightarrow$ $(a-\delta, a+\delta)$ contains at least one point of S other than a $\Rightarrow$ Every del. NBD of a contains a point of S $\Rightarrow$ a is a limit point of S.

Since any point of I is a limit point of S, S is a dense subset of I

3. (b) If S is a dense subset of I and $a \in I$, prove that there is a sequence $\{x_n\}$ of points in S such that $x_n \to a$ as $n \to \infty$

pf. Suppose S is a dense subset of I. Then every point of I is a limit point of S.

Let $a \in I$, then a is a limit point of S

* $\Rightarrow$ every NBD of a contains a point of S

Let x_n be a point in $\delta = \frac{1}{n}$ NBD of a with $x_n \in S$

Then $x_n \to a$ as $n \to \infty$

3(c) Suppose f, g are continuous functions on I that agree on a dense subset S of I, that is, $f(x) = g(x)$ for $x \in S$. Prove that $f(x) = g(x)$ for all $x \in I$

See HW#5 (?)

pf. Let x be any point in I. Since S is a dense subset of I, x is a limit point of S. Then $\exists$ a sequence $\{x_n\}$ of points in S $\ni$ $x_n \to x$ as $n \to \infty$. Since f, g are continuous on I, f, g are continuous at x

by (b),

$$\Rightarrow f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$$

since $x_n \in S$

$\therefore$ $f(x) = g(x)$ for all $x \in I$